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The Higher Flows of Harmonic Maps

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The Higher Flows of Harmonic Maps

by

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The Higher Flows of Harmonic Maps

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This thesis relates the AKNS-hierarchy of flows, called positive flows, to the hierarchy of flows that include the harmonic map equation, called negative flows. I show that these two hierarchies are actually part of one large hierarchy of commuting flows. This is done by looking at the loop group of two circles into a Lie group and determining the projections in a algebra splitting. The projections are then applied to the action of the Virasoro algebra.

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Chapter 1

Introduction

The motivating example for this thesis is the harmonic map equation for maps $\phi : \mathbb{R}^{1,1} \to G$, where $\mathbb{R}^{1,1}$ is the Lorentz space and G is a real matrix Lie group. The case of maps $\phi : \mathbb{R}^2 \to G$ has been thoroughly studied, where as the Lorentz case has not received the same attention. This is partially due to the nature of the reality condition that arises in the loop group in the elliptic case. The reality condition gives a true factorization of the loop group into two subgroups. This was shown by McIntosh in [3]. Unfortunately, this is not known to be true in the Lorentz case.

In a different gauge, the harmonic map equation can be seen as a flow on a solution space. It is actually the first in a hierarchy of flows, called the negative flows. This hierarchy is indexed by the negative integers, of which, the harmonic map flow is the -1 flow. The question still remained to be asked, "Do these flows commute?"

There is also a hierarchy of positive flows called the AKNS hierarchy. This hierarchy is known to contain the Non-linear Shrödinger equation, the mKdV equation and the n-wave equation. It is also known that these flows arise from a right dressing action and they commute with each other. There is also a way to fit the -1 flow into the hierarchy of positive flows.

Both the positive and negative flows use the same loop group formulation and factorization of LG into L_+G, the subgroup of maps that extend holomorphically around 0, and L_-G, the subgroup of maps that extend holomorphically around ∞. Following the work of Uhlenbeck in [10], Guest uses the loop group denoted as $\mathbb{L}G$ to treat the elliptic harmonic map equation in the original gauge in [2]. However, Guest uses different techniques than Uhlenbeck and Terng.

In this this thesis, I determine the Lie algebra projections in the factorization of $\mathbb{L}G$. I then use the techniques of Uhlenbeck and Terng to show that the negative and positive flows are actually part of one large hierachy and all of the flows commute. This all can be done in the harmonic gauge.

Finally, I apply the projections on the loop sub-algebras to a paper of Uhlenbeck and Vajiac [11]. The indirect half-Virasoro action on the extended harmonic map is seen to be equivalent to the direct half-Virasoro action on the scattering data for $v(\lambda) = \lambda^k, k \geq 2$.

Chapter 2

The Harmonic Map Equation

Let $\phi : \mathbb{R}^{1,1} \to G$ be a smooth map from Lorentz space to a compact Lie group. While the results of this paper hold for any compact real Lie group, all proofs are for SU_n. Define the functional

$$E(\phi) = \frac{1}{2}\int \left|\phi^{-1}\phi_x\right| + \left|\phi^{-1}\phi_t\right| dxdt.$$

The Euler-Lagrange equation for E is

$$\frac{\partial}{\partial t}(\phi^{-1}\phi_t) - \frac{\partial}{\partial x}(\phi^{-1}\phi_x) = 0,$$

and its solutions are said to solve the harmonic map equation. A change to characteristic coordinates, $\xi = t + x$ and $\eta = t - x$, gives the following differential equation

$$\frac{\partial}{\partial \eta}(\phi^{-1}\phi_\xi) + \frac{\partial}{\partial \xi}(\phi^{-1}\phi_\eta) = 0. \tag{2.1}$$

The goal is to have a Lax pair formulation of the harmonic map equation. Let $A = \phi^{-1}\phi_\xi$ and $B = \phi^{-1}\phi_\eta$. Equation (2.1) is equivalent to

$$A_\eta + B_\xi = 0.$$

Next, given $A, B : \mathbb{R}^{1,1} \to \mathfrak{g}$, we can ask, when does there exist a $\phi : \mathbb{R}^{1,1} \to G$ such that $\phi^{-1}\phi_\xi = A$ and $\phi^{-1}\phi_\eta = B$? The answer is in the following well known theorem.

Proposition 2.1. [2] *Let G be a Lie group. Let $\alpha = Ad\xi + Bd\eta$, where $A, B : \mathbb{R}^{1,1} \to \mathfrak{g}$. The following statements are equivalent:*

$(*)$ *α satisfies the equation $d\alpha + \dfrac{1}{2}[\alpha \wedge \alpha] = 0$ i.e., $A_\eta - B_\xi = [A, B]$*

$(**)$ *There exists a map $F : \mathbb{R}^{1,1} \to G$ such that $\alpha = F^{-1}dF$*

Therefore, the harmonic map equation is equivalent to the following pair of equations:

$$\begin{aligned} &A_\eta + B_\xi = 0 \\ &A_\eta - B_\xi = [A, B] \end{aligned} \tag{2.2}$$

Let $\lambda \in \mathbb{C}^*$ and let $A_\lambda = \frac{(1-\lambda^{-1})}{2}A$ and $B_\lambda = \frac{(1-\lambda)}{2}B$. Equation (2.2) is equivalent to

$$(A_\lambda)_\eta - (B_\lambda)_\xi = [A_\lambda, B_\lambda], \text{ for all } \lambda \in \mathbb{C}^*. \tag{2.3}$$

This is seen by comparing the coefficients of λ^i, $i = -1, 0, 1$ in equation (2.3):

$$\begin{aligned} \lambda^1 : &\quad A_\eta = \frac{1}{2}[A, B] \\ \lambda^{-1} : &\quad B_\xi = \frac{1}{2}[B, A] \\ \lambda^0 : &\quad A_\eta - B_\xi = [A, B] \end{aligned} \tag{2.4}$$

If $A, B \in \mathfrak{g}$, then $A_\lambda, B_\lambda \in \mathfrak{g} \otimes \mathbb{C}$. If G is a real Lie group, we need to enforce a reality condition on A_λ and B_λ. In order that $A_\lambda, B_\lambda \in \mathfrak{g}$, it is required that $c(A_\lambda) = A_\lambda$ and $c(B_\lambda) = B_\lambda$, where $c(X)$ is conjugation with respect to $\mathfrak{g}$. For $G = SU_n$, $c(X) = -X^*$ and we need $-A^*_\lambda = A_\lambda$ and $-B^*_\lambda = B_\lambda$. This implies if

$\lambda = \bar{\lambda}$, then $A_\lambda, B_\lambda \in \mathfrak{g}$.

Equation (2.3) is the Lax pair formulation of the harmonic map equation. To see this, by proposition (2.1), if equation (2.3) holds for some $A, B \in \mathfrak{g}$, then there exists a function $F : \mathbb{C}^* \times \mathbb{R}^{1,1} \to G$ such that

$$\begin{aligned} F^{-1}F_\eta &= A_\lambda, \\ F^{-1}F_\xi &= B_\lambda. \end{aligned} \tag{2.5}$$

If we evaluate F at $\lambda = -1$, we have

$$\begin{aligned} F^{-1}F_\eta &= A, \\ F^{-1}F_\xi &= B, \end{aligned}$$

and by equation (2.2), $F(\xi, \eta, -1)$ is harmonic. If we start with a harmonic map ϕ, then there exists an $F : \mathbb{R}^{1,1} \times \mathbb{C}^* \to G_{\mathbb{C}}$ with $F(\xi, \eta, -1) = \phi(\xi, \eta)$. In the following chapters, we will determine a loop group formulation of the harmonic map equation. This will allow us to use loop group techniques to find a hierarchy of commuting flows for the harmonic map equation.

Chapter 3

Loop Groups and Loop Algebras

This section sets up the necessary notation and concepts involving loop groups.

Definition 3.1. *Let G be a matrix Lie group (real or complex). Define the loop group LG as follows:*

$$LG = \{\gamma \mid \gamma : S^1 \to G, \gamma \text{ is smooth}\}. \tag{3.1}$$

In words, it is the group of all smooth maps from the circle S^1 to the group G. Loop groups can consist of more than just smooth maps. Presley and Segal treat many cases of loop groups in [4]. They also treat the ∞-dimensional manifold properties and representation theory of loop groups, all of which will not be needed here.

The Riemann-Hilbert problem will be of fundemental interest to us.

Definition 3.2. Reimann-Hilbert problem. *Let Γ be a (not necessarily connected) simple closed contour in the Riemann sphere $\mathbb{C} \cup \infty$. Let F be a smooth matrix-valued function on Γ. When can we find matrix-valued functions F_+, F_- such that*

1. $F = F_-|_\Gamma F_+|_\Gamma$

2. *$F_+(F_-)$ is holomorphic on the interior (exterior) of Γ?*

In our first example, let $\gamma = \{|\lambda| = 1\}$. We define the interior to be $\Omega_+ = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ and the exterior is $\Omega_- = \{\lambda \in \mathbb{C} \cup \infty \mid |\lambda| \geq 1\}$. The Riemann-Hilbert problem can be restated by making the following definitions:

$$L_+G = \{\gamma \in LG \mid \gamma \text{ extends holomorphically to } \Omega_+\},$$
$$L_-G = \{\gamma \in LG \mid \gamma \text{ extends holomorphically to } \Omega_-\}.$$

The Riemann-Hilbert problem now asks, "Given a function $F \in LG$, when can we find functions $F_- \in L_-G$ and $F_+ \in L_+G$ such that $F = F_-F_+$ on $\gamma = S^1$?" If $G = GL_n\mathbb{C}$ then this can be done for generic $F \in LGL_n\mathbb{C}$ since the product $L_-GL_n\mathbb{C} \star L_+GL_n\mathbb{C}$ is an open dense subset of $LGL_n\mathbb{C}$.

Another, more complex, Riemann-Hilbert problem is obtained by setting

$$\Gamma = \gamma_1 \cup \gamma_2, \text{ where } \gamma_1 = \{\lambda \mid |\lambda| = \epsilon\} \text{ and } \gamma_2 = \{\lambda \mid |\lambda| = \epsilon^{-1}\}$$

for small ϵ. This loop group consists of smooth maps from $\gamma_1 \cup \gamma_2$ into a Lie group G and is denoted $\mathbb{L}G$ (in this paper only). Following the notation in [2], we define the exterior of γ to be $\Omega_E = \{\lambda \in \mathbb{C} \mid \epsilon \leq |\lambda| \leq \epsilon^{-1}\}$ and the interior as $\Omega_I = \{\lambda \in \mathbb{C} \cup \infty \mid |\lambda| \leq \epsilon \text{ and } |\lambda| \geq \epsilon^{-1}\}$. Then, the two loop groups needed for factorization are

$$\mathbb{L}_E G = \{f \in \mathbb{L}G \mid f \text{ extends holomorphically to } \Omega_E\},$$
$$\mathbb{L}_I G = \{f \in \mathbb{L}G \mid f \text{ extends holomorphically to } \Omega_I\}.$$

The factorization of $\mathbb{L}G$ is more complicated than that of LG. For $G_{\mathbb{C}}$ a complexification of a real Lie group $G_{\mathbb{R}}$, we look at the subgroup $\mathbb{L}^{\mathbb{R}}G_{\mathbb{C}}$ and require the reality condition $f^*(\bar{\lambda}^{-1}) = f(\lambda)^{-1}$, then in [3] McIntosh shows that there is a true factorization

$$\mathbb{L}^{\mathbb{R}}G_{\mathbb{C}} = \mathbb{L}_I^{\mathbb{R}}G_{\mathbb{C}}\mathbb{L}_E^{\mathbb{R},1}G_{\mathbb{C}}.$$

The superscript ^{1} implies that $f(1) = I$ for all $f \in \mathbb{L}_E^{\mathbb{R},1}G_{\mathbb{C}}$. Variations of this loop group will be used later in the paper.

The next step is to define the dressing actions. Let $G = G_1G_2$ where $G_1 \cap G_2 = Id$, then we can define the left dressing action of G_2 on G_1.

Definition 3.3. *Let G, G_1, G_2 be as above and let $g_1 \in G_1, g_2 \in G_2$. Define the left dressing action of G_2 on G_1 as follows:*

$$g_2 * g_1 = \tilde{g_1}, \text{ where } g_2g_1 = \tilde{g_1}\tilde{g_2},$$

for $\tilde{g_1} \in G_1$ and $\tilde{g_2} \in G_2$.

Just to explain further, we know that any element $g \in G$ can be written uniquely as $g = h_1h_2$ for $h_i \in G_i$. Therefore, we take $g = g_2g_1$ and "write it in the other order". Then, let $\tilde{g_1}$ be the part that is in G_1. The dressing action is a group action, i.e., $(g_2h_2) * g_1 = g_2 * (h_2 * g_1)$.

Along with a left dressing action, there is a right dressing action of G_1 on G_2. As usual, right actions involve inverses. Again, the right dressing action is a group action.

Definition 3.4. *Let G, G_1, G_2, g_1, g_2 be as above. The right dressing action of G_1 on G_2 is defined as:*

$$g_2 *_r g_1 = \tilde{g_2} \text{ where } g_2^{-1}g_1 = \tilde{g_1}\tilde{g_2}^{-1}.$$

Another useful technique is to expand an element $f \in LG$ in a power series.

For instance, a general $f \in LG$ will have the form

$$f = \cdots + A_{-2}\lambda^{-2} + A_{-1}\lambda^{-1} + A_0 + A_1\lambda^1 + A_2\lambda^2 + \cdots = \sum_{-\infty}^{\infty} A_i\lambda^i,$$

where $A_i \in M_n$. Some other loop group expansions are as follows:

$$L_+G = \{f = A_0 + A_1\lambda + \cdots = \sum_{i=0}^{\infty} A_i\lambda^i\},$$
$$L_-G = \{f = \cdots + A_{-1}\lambda^{-1} + A_0 = \sum_{i=0}^{\infty} A_i\lambda^{-i}\},$$
$$L_-^{\infty}G = \{f = \cdots + A_{-2}\lambda^{-2} + A_{-1}\lambda^{-1} = \sum_{i=1}^{\infty} A_i\lambda^{-i}\}.$$

The expansions for $\mathbb{L}G$ and its subgroups will be discussed later.

Just as finite dimensional Lie groups have Lie algebras, the infinite dimensional loop groups have Lie algebras, called loop algebras.

Definition 3.5. *The Lie algebra to a loop group LG is defined to be*

$$L\mathfrak{g} = \{f : S^1 \to \mathfrak{g} \mid f \text{ is smooth}\}.$$

It should be noted that a splitting of loop algebras does not necessarily lead to a true factorization of Lie groups. For instance, $L\mathfrak{g} = L_+\mathfrak{g} \oplus L_-^{\infty}\mathfrak{g}$, but as mentioned before, $L_+G * L_-^{\infty}G$ is only a dense open subset of LG. When our group does not have a true factorization our calculations are only true where the factorization exists. This is the norm rather than the exception in this paper. However, the subset of the loop groups which admits a factorization is open. Hence the infinitesimal formulas are always valid.

Chapter 4

The Positive Flows

The following treatment of the positive flows can be found in [5]. First, assume $a \in \mathfrak{sl}_n\mathbb{C}$ is a diagonal matrix with distinct eigenvalues. Define the following spaces:

$$\mathfrak{sl}_n\mathbb{C}_a = \{y \in \mathfrak{sl}_n\mathbb{C} \mid [a, y] = 0\},$$
$$\mathfrak{sl}_n\mathbb{C}_a^\perp = \{\xi \in \mathfrak{sl}_n\mathbb{C} \mid \ <\xi, y>= 0 \ \forall y \in \mathfrak{sl}_n\mathbb{C}_a\}.$$

If we define $< x, y >= tr(xy)$ as the bilinear form on $\mathfrak{sl}_n\mathbb{C}$, then $\mathfrak{sl}_n\mathbb{C}_a$ is the space of all diagonal matricies in $\mathfrak{sl}_n\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}_a^\perp$ is the space of all off diagonal matricies in $\mathfrak{sl}_n\mathbb{C}$.

The following theorem and its proof can be found in [5]. The theorem is originally attributed to Ablowitz-Kaup-Newell–Segur for $n = 2$ in [1], and for all n by Satinger in [7].

Theorem 4.1. [1][5][7] *Let $a \in \mathfrak{sl}_n\mathbb{C}$ be as above. Then for each $b \in \mathfrak{sl}_n\mathbb{C}_a$. there exists a sequence of smooth functions $Q_{b,j}$ satisfying the following conditions:*

1. $Q_{b,0} = b$,

2. $(Q_{b,j})_x + [u, Q_{b,j}] = [Q_{b,j+1}, a]$, *where* $u = Q_{a,1}$,

3. the asymptotic expansion

$$tr\left(\sum_{j=0}^{\infty} Q_{b,j}\lambda^{-j}\right)^k \sim tr(b^k)$$

for all $1 \le k \le n$.

Moreover, $Q_{b,j}$*, can be solved using* $1-3$ *and is a polynomial in* $u, d_x u, \ldots, d_x^{j-1}u$.

Since $Q_{b,j}$ is a polynomial in u and its dervatives, we make the following definition.

Definition 4.1. *The* j*th-flow equation on* $C(\mathbb{R}, \mathfrak{sl}_n\mathbb{C}_a^{\perp})$ *defined by* b *is:*

$$u_t = (Q_{b,j})_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a] \tag{4.1}$$

The jth-flow has a Lax pair:

$$\left[\frac{\partial}{\partial x} + u, \frac{\partial}{\partial t} + Q_{b,j}\right] = 0. \tag{4.2}$$

In certain cases, the jth-flows can be computed.

Example: Take $n = 2$ and $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, then

$$\mathfrak{sl}_2\mathbb{C}_a^{\perp} = \left\{ \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \;\middle|\; q, r \in \mathbb{C} \right\}.$$

Also, if we write

$$Q_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix},$$

we can use part 2 of theorem (4.1) to determine the entries of Q_n. First, look at the off diagonal entries of

$$(Q_n)_x + [u, Q_n] = [Q_{n+1}, a].$$

We get

$$\begin{aligned} B_{n+1} &= \frac{i}{2}((B_n)_x - 2A_n q), \\ C_{n+1} &= -\frac{i}{2}((C_n)_x + 2A_n r). \end{aligned} \tag{4.3}$$

The first few terms are given by the assumptions. $Q_0 = \mathrm{diag}(i, -i)$ and $A_1 = 0, B_1 = q, C_1 = r$. Plugging these into equation (4.3), we get

$$B_2 = \frac{i}{2} q_x, \quad C_2 = -\frac{i}{2} r_x.$$

We use condition 2 in theorem (4.1) to get A_2:

$$tr((a + u\lambda^{-1} + Q_2\lambda^{-2} + \cdots)^2) \sim tr(a^2).$$

From the coefficient of λ^{-2} we obtain

$$tr(2aQ_2 + u^2) = 0$$

or $A_2 = \frac{i}{2} qr$. Thus, the first three Q_i's are:

$$\begin{aligned} Q_{a,1}(u) &= u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \\ Q_{a,2}(u) &= \frac{i}{2} \begin{pmatrix} qr & q_x \\ -r_x & -qr \end{pmatrix} \\ Q_{a,3}(u) &= \frac{1}{4} \begin{pmatrix} qr_x - rq_x & -q_{xx} + 2q^2 r \\ -r_{xx} + 2qr^2 & -qr_x + rq_x \end{pmatrix}. \end{aligned}$$

Therefore the first three flows in the $SL_2\mathbb{C}$ hierarchy defined by a are

$$
\begin{aligned}
q_t &= q_x, \quad r_t = r_x \\
q_t &= \frac{i}{2}(q_{xx} - 2q^2 r), \quad r_t = \frac{i}{2}(-r_{xx} + 2qr^2), \\
q_t &= \frac{1}{4}(-q_{xxx} + 6qrq_x), \quad r_t = \frac{1}{4}(-r_{xxx} + qrr_x).
\end{aligned}
$$

Notice, if we restrict to $u \in \mathfrak{su}^{\perp}_{(2)a}$; i.e. $r = -q^*$, then the second flow is the Non-linear Schrödinger equation and the third flow is the modified KdV equation.

We are finally ready to show that the positive flows arise from a right dressing action. This concept is fundamental to the rest of the paper. The factorization will use the subgroups $L^{\infty}_{-}SL_n\mathbb{C}$ and $L_{+}SL_n\mathbb{C}$. Let a, b be as before. We will consider the right dressing action of $e^{a\lambda x + b\lambda^j t} \in C^{\infty}(\mathbb{R}^{1,1}, L_{+}SL_n\mathbb{C})$ on $f \in L^{\infty}_{-}SL_n\mathbb{C}$. Define $M(x,t,\lambda)$ as the right dressing action:

$$M(x,t,\lambda) \equiv f *_r e^{a\lambda x + b\lambda^j t}.$$

If we write down $M(x,t,\lambda)$ using the definition of a right dressing action (definition 3.4), we have

$$f^{-1} e^{a\lambda x + b\lambda^j t} = F(x,t,\lambda) M(x,t,\lambda)^{-1} \tag{4.4}$$

where $F(x,t,\lambda) \in C^{\infty}(\mathbb{R}^{1,1}, L_{+}SL_n\mathbb{C})$. Next, expand $M(x,t,\lambda)$ around $\lambda = \infty$ to get

$$
\begin{aligned}
M(x,t,\lambda)(\lambda) &= I + m_1(x,t)\lambda^{-1} + m_2(x,t)\lambda^{-2} + \cdots \\
M(x,t,\lambda)^{-1} &= I - m_1(x,t)\lambda^{-1} + \cdots
\end{aligned}
\tag{4.5}
$$

Also, for $X \in \mathfrak{sl}_n\mathbb{C}$ (later, X will be a or b) we have

$$M^{-1} X M = I + \xi_{X,1}\lambda^{-1} + \xi_{X,2}\lambda^{-2} + \cdots \tag{4.6}$$

where $\xi_{X,1}$ can be expicitly computed to be $\xi_{X,1} = [X, m_1(x,t)]$. The following theorem by Terng in [5] shows that this dressing action gives rise to the positive flows.

Theorem 4.2. [5] *Let a, b, F, m, u and $\xi_{b,j}$ be as above. Then*

1. *$\xi_{b,j} = Q_{b,j}$ as constructed in Theorem (4.1)*
2. *the map $u(x,t) = [a, m_1(x,t)]$ is a solution of the jth flow.*

While the proof is illuminating, we will only briefly discuss the steps here. In particular, part one is proved by showing the $\xi_{b,j}$ satisfy the same differential equation as $Q_{b,j}$, namely, $(\xi_{b,j})_x + [u, \xi_{b,j}] = [\xi_{b,j+1}, a]$. By uniqueness, $\xi_{b,j} = Q_{b,j}$. Part 2 is proved by showing the connection

$$\theta_\lambda = (a\lambda + u)dx + \left(\sum_{i=0}^{j} Q_{b,j-i}(u)\lambda^{j-i} \right) dt$$

is flat for all $\lambda \in \mathbb{C}$ and using the following theorem.

Theorem 4.3. [5] *Let $\mathbf{O}$ be an open subset of $\mathbb{R}^{1,1}$, and $u : \mathbf{O} \to \mathfrak{sl}_n\mathbb{C}_a^\perp$ a smooth map. Then u is a solution of (4.1) if and only if*

$$\theta_\lambda \equiv (a\lambda + u)dx + (b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u))dt$$

is a flat connection for all $\lambda \in \mathbb{C}$.

Finally, Terng shows that the jth flow is generated by the element $b\lambda^j$ and that the flows commute. Let A be the space of all diagonal matricies in $\mathfrak{sl}_n\mathbb{C}$. We look at the action of L_+A on the phase space $C(\mathbb{R}, \mathfrak{sl}_n\mathbb{C}_a^\perp)$ of the flows in the $SL_n\mathbb{C}$-hierarchy. Given $f \in L_-SL_n\mathbb{C}$, define

$$M_f(x,\lambda) = f *_r e^{a\lambda x}.$$

Expand $M_f(x,\lambda)$ around $\lambda=\infty$ to get

$$M_f(x,\lambda) = I + (m_f)_1(x)\lambda^{-1} + \cdots .$$

The dressing action is only a local map, so we need to introduce $C_0^\infty(\mathbb{R}, \mathfrak{sl}_x\mathbb{C}_a^\perp)$, the space of germs of smooth maps from $\mathbb{R}$ to $\mathfrak{sl}_n\mathbb{C}_a^\perp$ at 0. Define the map

$$\Psi : L_-SL_n\mathbb{C} \to C_0(\mathbb{R}, \mathfrak{sl}_n\mathbb{C}_a^\perp)$$

to be

$$\Psi(f) = [a, (m_f)_1].$$

The following proposition says that $Im(\Psi)$ is the homogenous space $L_-SL_n\mathbb{C}/L_-A$.

Proposition 4.1. [5]

1. *If $f \in L_-SL_n\mathbb{C}$, then $\Psi(f) = 0$ iff $f \in L_-A$.*
2. *If $f, g \in L_-SL_n\mathbb{C}$, then $\Psi(f) = \Psi(g)$ iff $fg^{-1} \in L_-A$.*

Now, $h \in L_+A$ acts on $Im(\Psi)$ by

$$h * \Psi(f) = \Psi(f *_r h).$$

If we let $X_{b,j}$ denote the vector field on $Im(\Psi)$ defined by $b\lambda^j$, then theorem (4.2) implies

$$X_{b,j} = (Q_{b,j})_x + [u, Q_{b,j}].$$

Therefore, the flow is the jth flow. We also get that the flows commute since L_+A is abelian.

A quick note on the notation. It is unfortunate, but in the literature, x and t are used for characteristic coordinates. This notation will be used here to avoid confusion with other papers. Therefore, for the rest of the paper, $x = \eta$ and $t = \xi$. Since we will not use laboratory coordinates again, this should not cause any confusion.

Chapter 5

The Negative Flows

The negative flows, which include the harmonic map equation, are induced from a local left dressing action. The negative flows are not differential equations in u and its derivatives as are the positive flows. Terng and Uhlenbeck treat the decay case of the negative flows in [6]. The following will be a formulation of the negative flows in the asymptotically constant case.

Definition 5.1. *Let a be a diagonal element in $\mathfrak{su}_n$. Define*

$$\mathfrak{U}_a = \{y \in \mathfrak{su}_n \mid [y, a] = 0\}$$
$$\mathfrak{U}_a^{\perp} = \{z \in \mathfrak{su}_n \mid < z, \mathfrak{U}_a >= 0\}.$$

Definition 5.2. *Given a Vector space V, define $S(\mathbb{R}, V)$ to be the space of all maps from $\mathbb{R}$ to V that are in the Schwartz class.*

Let $A : \mathbb{R} \to L_{+}\mathfrak{g}$ be defined as $A(x)(\lambda) = a\lambda + u(x)$ where $u(x) \in S(\mathbb{R}, \mathfrak{su}_n)$.

Definition 5.3. *The frame of $A = a\lambda + u(x)$ normalized at $x = 0$ is the smooth* *solution* $F(A) : \mathbb{R} \to L_+SU(n)$ *of*

$$F^{-1}F_x = A \qquad \lim_{x\to-\infty} F(x,\lambda)e^{-a\lambda x} = I,$$
$$F(0,\lambda) = I.$$

Given $b \in \mathfrak{su}_n$ which commutes with a, we can expand the expression $F^{-1}bF \in L_+\mathfrak{g}$ around $\lambda = 0$ to get

$$F^{-1}bF = \xi_0 + \xi_1\lambda + \xi_2\lambda^2 + \cdots.$$

Using the definition of F, we see that

$$(F^{-1}bF)_x + [A, F^{-1}bF] = 0. \tag{5.1}$$

Comparing the coefficients of λ^i in equation (5.1) we get

$$\begin{cases} (\xi_0)_x + [u, \xi_0] = 0 \\ (\xi_i)_x + [u, \xi_i] + [a, \xi_{i-1}] = 0. \end{cases}$$

We can solve the ξ_i's explicitly in terms of a and u. Let $g : \mathbb{R} \to GL_n(\mathbb{C})$ be the solution to

$$\begin{cases} g^{-1}g_x = u \\ \lim_{x\to-\infty} g(x) = I. \end{cases} \tag{5.2}$$

Then

$$\begin{aligned} &\xi_0 = g^{-1}bg \\ &\xi_i(x) = -g^{-1}(x)\left(\int_{-\infty}^{x} g(y)[a, \xi_{i-1}]g^{-1}(y)dy\right)g(x). \end{aligned} \tag{5.3}$$

We can form a Lax pair for this system as follows

$$\left[\frac{\partial}{\partial x} + A, \frac{\partial}{\partial t} + (F^{-1}b\lambda^{-m}F)_-\right] = 0. \tag{5.4}$$

In order for (5.4) to be true, we need the following to hold

$$\begin{aligned} \frac{da}{dt} &= 0, \\ \frac{du}{dt} &= [a, \xi_{m-1}]. \end{aligned} \tag{5.5}$$

Equation (5.5) is called the -m-flow. The -1-flow is the harmonic map equation as we see from the following proposition.

Proposition 5.1. [6] *Fix diagonal elements a, b of $\mathfrak{su}_n$. Suppose $u(x,t)$ is a solution of the -1-flow. There is a unique solution $F(x,t,\lambda)$ for*

$$\begin{aligned} &F^{-1}F_x = a\lambda + u, \\ &F^{-1}F_t = \lambda^{-1}g^{-1}bg, \\ &F_\lambda(0,0) = I. \end{aligned}$$

Set $s(x,t) = F(x,t,-1)F(x,t,1)^{-1}$. Then $s : \mathbb{R}^{1,1} \to SU_n$ is a harmonic map, i.e.

$$\left(s^{-1}s_x\right)_t + \left(s^{-1}s_t\right)_x = 0.$$

Chapter 6

Loop Groups: Revisited

In order to put the positive and negative flows in the same framework, we introduce a more complex loop group. This is the loop group usually associated with harmonic maps and consists of maps from two circles into a (real or complex) Lie group G. The factorization of this loop group combines the positive and negative flow equations.

Let $0 < \epsilon \ll 1$, $\gamma_1 = \{\lambda \in \mathbb{C} \mid |\lambda| = \epsilon\}$ and $\gamma_2 = \{\lambda \in \mathbb{C} \mid |\lambda| = \epsilon^{-1}\}$. Define $\Gamma = \gamma_1 \cup \gamma_2$ and

$$\mathbb{L}G = \{f : \Gamma \to G \mid f \text{ is smooth}\}.$$

We use the following factorization: Let $E = \{\lambda \in \mathbb{C} \mid \epsilon \leq |\lambda| \leq \epsilon^{-1}\}$ and $I = I_1 \cup I_2$ where $I_1 = \{\lambda \in \mathbb{C} \mid |\lambda| \leq \epsilon\}$ and $I_2 = \{\lambda \in \mathbb{C} \mid |\lambda| \geq \epsilon^{-1}\}$. This set up can be seen in figure 6.1 on the next page.

Returning to loop groups, we have

$$\begin{aligned}\mathbb{L}_E G =& \{f \in \mathbb{L}G \mid f \text{ extends holomorphically to } E\}\\ \mathbb{L}_I G =& \{f \in \mathbb{L}G \mid f \text{ extends holomorphically to } I\}.\end{aligned}$$

The group $\mathbb{L}_E G$ will play the role of the positive group from the previous sections. However, the terms "positive" and "negative" become misleading as both

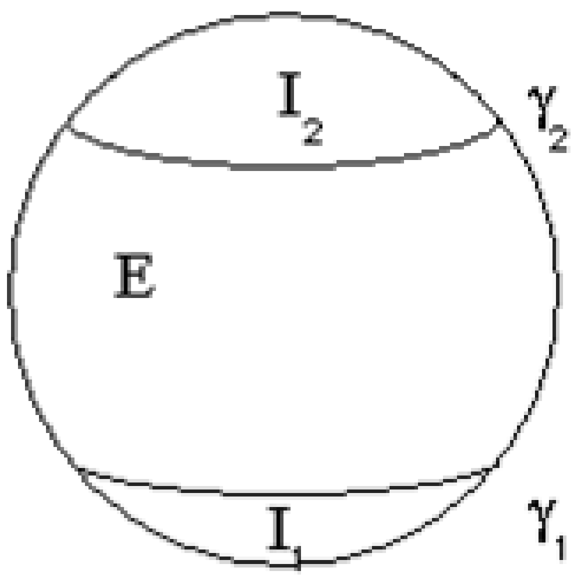

Figure 6.1: .

$\mathbb{L}_E G$ and $\mathbb{L}_I G$ have positive and negative powers of λ in the power series expansion. For this reason, we will refer to $\mathbb{L}_E G$ as the "E" group and $\mathbb{L}_I G$ as the "I" group. Also, the terms "positive" and "negative" will always refer to our original factorization.

As stated before, the reality condition for the elliptic harmonic map,

$$f^*(\bar{\lambda}^{-1}) = f^{-1}(\lambda),$$

gives a true factorization as shown by McIntosh [3]. This is a consequence of the map $\lambda \mapsto \bar{\lambda}^{-1}$ which takes $\gamma_1 \mapsto \gamma_2$. McIntosh uses this and the previous factorization used in chapter (3) to prove the factorization for loop groups with the elliptic reality condition.

Unfortunately, we cannot take advantage of this factorization when using the wave harmonic map reality condition $f^*(\bar{\lambda}) = f^{-1}(\lambda)$. Therefore, all computations are formal, i.e. they are true only where the factorization exists.

When the reality condition is introduced, we switch to the complexication, $G_{\mathbb{C}}$, of the real Lie group.

We now have

$$\begin{aligned}
\mathbb{L}^{\mathbb{R}} G_{\mathbb{C}} &= \{f \in \mathbb{L}G \mid f^*(\bar{\lambda}) = f^{-1}(\lambda)\}, \\
\mathbb{L}^{\mathbb{R}}_E G_{\mathbb{C}} &= \{f \in \mathbb{L}_E G_{\mathbb{C}} \mid f^*(\bar{\lambda}) = f^{-1}(\lambda)\}, \\
\mathbb{L}^{\mathbb{R}}_I G_{\mathbb{C}} &= \{f \in \mathbb{L}_I G_{\mathbb{C}} \mid f^*(\bar{\lambda}) = f^{-1}(\lambda)\}.
\end{aligned}$$

A quick word on notation. First, the reality condition allows us to work with complex Lie groups and Lie algebras. A restriction on λ will force the functions to remain in the real Lie group. For instance, if $g \in L^{\mathbb{R}}_E SL_n\mathbb{C}$, and F is the extension of g to E, then $F(\lambda) \in SU_n$ for all $\lambda \in \mathbb{R}^* \cap E$.

Second, A map $g \in \mathbb{L}^{\mathbb{R}} G_{\mathbb{C}}$ is actually two maps $g(\lambda) = (g_1(\lambda), g_2(\lambda))$ where g_i

is defined on γ_i. The multiplication is component wise, so if $h(\lambda) = (h_1(\lambda), h_2(\lambda))$, then

$$(gh)(\lambda) = (g_1(\lambda)h_1(\lambda), g_2(\lambda)h_2(\lambda)).$$

A map $M(\lambda) \in \mathbb{L}_I^{\mathbb{R}} G_{\mathbb{C}}$ also consists two maps $M = (M_1, M_2)$ where M_1 extends holomorphically to I_1 and M_2 extends holomorphically to I_2. Finally, a map $F \in \mathbb{L}_E^{\mathbb{R}} G_{\mathbb{C}}$ is technically two maps $F = (F_1, F_2)$. Since it is holomorphic over a connected region, the expansion of F_1 is equal to the expansion of F_2. This means we can write $(F_1, F_1) = F$. This distinction is very important when multiplying group elements $(M_1, M_2)F \equiv (M_1, M_2)(F, F) = (M_1 F, M_2 F)$.

Also, when writing an element $M = (M_1, M_2)$ in the I group, we notice that the function M_1 extends holomorphically to a neighborhood around zero and hence has positive powers of λ. Also, M_2 extends around ∞ and has negative powers of λ. In order to delineate the coefficients of λ for an element in the I group, the following notation will be used:

$$M = (M_1, M_2) = (\xi_{0+} + \xi_{1+}\lambda + \xi_{2+}\lambda^2 + \cdots, \xi_{0-} + \xi_{1-}\lambda^{-1} + \xi_{2-}\lambda^{-2} + \cdots).$$

Just to reiterate, the number in the subscript refers to the power of λ and the $\pm$ refers to the sign of its power.

The right and left dressing actions are defined in the same way as before. However, this time the E group has a right action on the I group, and the I group has a left action on the E group. Let $M, f \in \mathbb{L}_I^{\mathbb{R},\infty} G_{\mathbb{C}}$ and $e, F \in \mathbb{L}_E^{\mathbb{R}} G_{\mathbb{C}}$, then

$$f *_r e = M \text{ where } f^{-1}e = FM^{-1}$$
$$f * e = F \text{ where } fe = FM.$$

In the next section, we will work out the explicit formulas for the Lie algebra projections. First, we will do a little exercise that gives some insight to a surprising feature of the projections. Let $(a, b) \in \mathbb{L}^{\mathbb{R}} G_{\mathbb{C}}$ and $(g, 1) \in \mathbb{L}_I^{\mathbb{R},\infty} G_{\mathbb{C}}$. If the factorization exists, we have

$$(c, d)(g, 1) = (e, e)(f_1, f_2),$$

for $(e, e) \in \mathbb{L}_E^{\mathbb{R}} G$ and $(f_1, f_2) \in \mathbb{L}_I^{\mathbb{R},\infty} G$. This gives the following equations

$$\begin{aligned} cg &= ef_1, \\ d &= ef_2. \end{aligned}$$

We notice that if the map (c, d) is fixed, then varying g forces f_2 to vary. Informally, this shows the "bottom cap" and the "top cap" affect each other. We will see this phenomenon explicitly at the Lie algebra level.

Chapter 7

Loop Algebras: Revisited

For this section, $G_{\mathbb{C}} = SL_n\mathbb{C}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n\mathbb{C}$. The "I" group that will be used in the formulation of the positive and negative flows is

$$\mathbb{L}_I^{\mathbb{R},\infty}G_{\mathbb{C}} = \{f = (f_1, f_2) \in \mathbb{L}_I^{\mathbb{R}}G_{\mathbb{C}} \mid f_2(\infty) = Id\}.$$

Let $\mathbb{L}_I^{\mathbb{R},\infty}\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $\mathbb{L}_I^{\mathbb{R},\infty}G_{\mathbb{C}}$ and $\mathbb{L}_E^{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $\mathbb{L}_E^{\mathbb{R}}G_{\mathbb{C}}$. We can write the power series expansions as:

$$\begin{aligned}
e &= \sum_{i\in\mathbb{Z}} A_i\lambda^i = \cdots + A_{-2}\lambda^{-2} + A_{-1}\lambda^{-1} + A_0 + A_1\lambda^1 + A_2\lambda^2 + \cdots \in \mathbb{L}_E^{\mathbb{R}}\mathfrak{g} \\
f &= (f_1, f_2) = (\sum_{i=0}^{\infty} B_{i+}\lambda^i, \sum_{i=1}^{\infty} B_{i-}\lambda^{-i}) \\
&= (B_{0+} + B_{1+}\lambda + B_{2+} + \cdots\ ,\ B_{1-}\lambda^{-1} + B_{2-}\lambda^{-2} + \cdots) \in \mathbb{L}_I^{\mathbb{R},\infty}\mathfrak{g}.
\end{aligned}$$

The reality condition demands that $A_i = -A_i^*$ and $B_{i\pm} = -B_{i\pm}^*$.

In order to apply the techniques of the positive flows we need to determine the projections onto $\mathbb{L}_I^{\mathbb{R},\infty}\mathfrak{g}_{\mathbb{C}}$ and $\mathbb{L}_E^{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$. This is done using Cauchy's integral formula, which picks out the holomorphic part of a function over the region integrated. Given

$g = (g_1, g_2) \in \mathbb{L}^{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$ we define the projections g_E and $g_I = (f_1, f_2)$ as follows:

$$\lambda \in E : g_E(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-\lambda}dz,$$
$$\lambda \in I_1 : f_1(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-\lambda}dz,$$
$$\lambda \in I_2 : f_2(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-\lambda}dz.$$

First, some notation:

Definition 7.1. *Let $h = (h_1, h_2) \in \mathbb{L}^{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$. Then h is of the form*

$$h = (\sum_{i\in\mathbb{Z}} A_i\lambda^i, \sum_{j\in\mathbb{Z}} B_j\lambda^j).$$

Define

$$h_{1_+} \equiv \sum_{i=0}^{\infty} A_i\lambda^i \qquad h_{1_-} \equiv \sum_{i=1}^{\infty} A_i\lambda^{-i},$$
$$h_{2_+} \equiv \sum_{j=0}^{\infty} B_j\lambda^j \qquad h_{2_-} \equiv \sum_{j=1}^{\infty} B_j\lambda^{-j}.$$

The explicit projections are as follows:

Theorem 7.1. *If $g = (g_1, g_2) \in \mathbb{L}^{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$, then*

$$g_E = g_{1_-} + g_{2_+}$$
$$g_I = (g_{1_+} - g_{2_+}, g_{2_-} - g_{1_-}).$$

Proof. The general idea will be to break up the integrals until we can reduce to the previous splitting in the $\pm$ case.

For $\lambda \in E$ and $\Gamma = \gamma_1 \cup \gamma_2$:

$$\begin{aligned}
g_E &= \frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2} \frac{g(z)}{z-\lambda}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{g(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{g(z)}{z-\lambda}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{g_1(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{g_2(z)}{z-\lambda}dz \\
&= g_{1_-} + g_{2_+}.
\end{aligned}$$

For $\lambda \in I_1$, we get:

$$\begin{aligned}
g_{I_1} &= -\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2} \frac{g(z)}{z-\lambda}dz \\
&= -\left(\frac{1}{2\pi i}\int_{-\gamma_1} \frac{g(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{g(z)}{z-\lambda}dz\right) \\
&= -\left(\frac{1}{2\pi i}\int_{-\gamma_1} \frac{g_1(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{g_2(z)}{z-\lambda}dz\right) \\
&= g_{1_+} - g_{2_+}.
\end{aligned}$$

Finally, for $\lambda \in I_2$, we have:

$$\begin{aligned}
g_{I_1} &= -\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2} \frac{g(z)}{z-\lambda}dz \\
&= -\left(\frac{1}{2\pi i}\int_{\gamma_1} \frac{g(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{-\gamma_2} \frac{g(z)}{z-\lambda}dz\right) \\
&= -\left(\frac{1}{2\pi i}\int_{\gamma_1} \frac{g_1(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{-\gamma_2} \frac{g_2(z)}{z-\lambda}dz\right) \\
&= g_{2_-} - g_{1_-}.
\end{aligned}$$

$\square$

The overall minus sign in the g_I integral is from looking at the limiting case in figure 7.1 on the next page. Notice that the I region is the exterior of one large loop, and hence gets a minus sign in its Cauchy Integral.

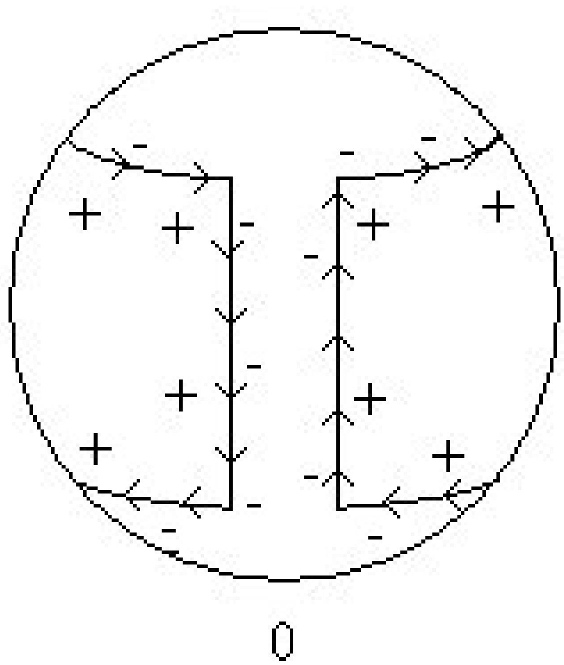

Figure 7.1: .

Chapter 8

The Higher Flows

Again in this section, we let $G_{\mathbb{C}} = SL_n\mathbb{C}$ and $G_{\mathbb{R}} = SU_n$. We use the more complicated loop $\mathbb{L}^{\mathbb{R}}SL_n\mathbb{C}$ and the subgroups $\mathbb{L}_I^{\mathbb{R},\infty}SL_n\mathbb{C}$, $\mathbb{L}_E^{\mathbb{R}}SL_n\mathbb{C}$. Let $f(\lambda) \in \mathbb{L}_I^{\mathbb{R},\infty}SL_n\mathbb{C}$ and a, b be diagonal elements in $\mathfrak{su}_n \subset \mathfrak{sl}_n\mathbb{C}$. The the right action of $e^{a\lambda x + b\lambda^j t}$ on $f(\lambda)$ for $j \neq 0$ is defined as

$$f(\lambda) *_r e^{a\lambda x + b\lambda^j t} = M(x, t, \lambda).$$

Here

$$f(\lambda)^{-1} e^{a\lambda x + b\lambda^j t} = F(x, t, \lambda) M(x, t, \lambda)^{-1},$$

with $F \in C^\infty(\mathbb{R}^{1,1}, \mathbb{L}_E^{\mathbb{R}}SL_n\mathbb{C})$ and $M \in C^\infty(\mathbb{R}^{1,1}, \mathbb{L}_I^{\mathbb{R},\infty}SL_n\mathbb{C})$. We notice that this loop group factorization allows negative powers of λ in the "E" group. This is in contrast to the positive group of the previous factorization.

By switching to the more complex loop group, we will see that both the positive and negative flows arise from the dressing action of $\mathbb{L}_E^{\mathbb{R}}SL_n\mathbb{C}$ on $\mathbb{L}_I^{\mathbb{R},\infty}SL_n\mathbb{C}$. This will also be used to show that the positive and negative flows commute.

Following the techniques of the positive flows, calculate $F^{-1}F_x$ and $F^{-1}F_t$:

$$
\begin{aligned}
F^{-1}F_x &= (M^{-1}e^{-a\lambda x - b\lambda^j t}f)(f^{-1}e^{a\lambda x + b\lambda^j t}M)_x \\
&= (M^{-1}e^{-a\lambda x - b\lambda^j t}f)(f^{-1}a\lambda e^{a\lambda x + b\lambda^j t}M + f^{-1}e^{a\lambda x + b\lambda^j t}M_x) \\
&= M^{-1}a\lambda M + M^{-1}M_x \\
&= (M^{-1}a\lambda M)_E,
\end{aligned}
$$

where the last line is true because $F^{-1}F_x \in \mathbb{L}_E^{\mathbb{R}}\mathfrak{sl}_n\mathbb{C}$. An identical computation for $F^{-1}F_t$ gives

$$F^{-1}F_t = (M^{-1}b\lambda^j M)_E.$$

If we write

$$M(x,t,\lambda) = (m_{0+} + m_{1+}\lambda + m_{2+}\lambda^2 \cdots, I + m_{1-}\lambda^{-1} + m_{2-}\lambda^{-2} \cdots),$$

then the first few terms of $M(x,t,\lambda)^{-1}$ are

$$M(x,t,\lambda)^{-1} = (m_{0+}^{-1} - m_{0+}^{-1}m_{1+}m_{0+}^{-1}\lambda + \cdots, I - m_{1-}\lambda^{-1} + \cdots).$$

The general form for $M^{-1}bM$ is

$$M^{-1}bM = (\xi_{0+} + \xi_{1+}\lambda + \xi_{2+}\lambda^2 \cdots, b + \xi_{1-}\lambda^{-1} + \xi_{2-}\lambda^{-2} \cdots). \tag{8.1}$$

However, we can do better and write it in terms of M:

$$M^{-1}XM = (m_{0+}^{-1}Xm_{0+} - [m_{0+}^{-1}m_{1+}, m_{0+}^{-1}Xm_{0+}]\lambda + \cdots, X + [X, m_{1-}]\lambda^{-1} + \cdots).$$

Next, we compute the projections $(M^{-1}a\lambda M)_E$ and $(M^{-1}b\lambda^j M)_E$ for $j > 0$.

$$
\begin{aligned}
(M^{-1}a\lambda M)_E =&(m_{0+}^{-1}am_{0+}\lambda - [m_{0+}^{-1}m_{1+}, m_{0+}^{-1}am_{0+}]\lambda^2 + \cdots, a\lambda + [a, m_{1-}] + \cdots)_E \\
=&a\lambda + [a, m_{1-}] \\
=&a\lambda + u.
\end{aligned}
$$

Here we define $u \equiv [a, m_{1-}]$ as before. Also, we have

$$
\begin{aligned}
(M^{-1}b\lambda^j M)_E =&(\xi_{0+}\lambda^j - \xi_{1+}\lambda^{j+1} + \cdots, b\lambda^j + \xi_{1-}\lambda^{j-1} + \cdots)_E \\
=&b\lambda^j + \xi_{1-}\lambda^{j-1} + \cdots + \xi_{(j-1)-}\lambda + \xi_{j-}.
\end{aligned}
$$

This new technique reproduces the positive flows. It still needs to be checked that each ξ_{i+} satisfies the appropriate differential equations. This is done in the same manner as before.

$$
\begin{aligned}
(M^{-1}bM)_x &= [M^{-1}bM, M^{-1}M_x] \\
&= [M^{-1}bM, F^{-1}F_x - M^{-1}a\lambda M] \\
&= [M^{-1}bM, a\lambda + u - M^{-1}a\lambda M] \\
&= [M^{-1}bM, a\lambda + u], \text{ since } [a, b] = 0.
\end{aligned}
\tag{8.2}
$$

Expanding $M^{-1}b\lambda^j M$ and comparing the coefficients of λ^i gives the differential equations for the positive flows:

$$
(\xi_{i-})_x = [\xi_{(i+1)-}, a] + [\xi_{i-}, u].
$$

As stated before, $e^{a\lambda x + b\lambda^{-j}t},\ j > 0$ is now a valid element of the "E" group. Repeating the same computations as before we get the equations:

$$
\begin{aligned}
F^{-1}F_x &= (M^{-1}a\lambda M)_E. \\
F^{-1}F_t &= (M^{-1}b\lambda^{-j}M)_E.
\end{aligned}
$$

By equation (8.1), we have

$$
\begin{aligned}
(M^{-1}b\lambda^{-j}M)_E &= (\xi_{0+}\lambda^{-j} - \xi_{1+}\lambda^{-j+1} + \cdots, b\lambda^{-j} + \xi_{1-}\lambda^{-j-1} + \cdots)_E \\
&= \xi_{0+}\lambda^{-j} - \xi_{1+}\lambda^{-j+1} + \cdots + \xi_{(j-1)+}\lambda^{-1}.
\end{aligned}
$$

Finally, to get the differential equations of the negative flows, we do the same computation as before and look at the terms that show up for $M^{-1}b\lambda^{-j}M$. ie $(M^{-1}bM)_x = [M^{-1}bM, a\lambda + u]$. This yields the negative flows:

$$
\begin{aligned}
(\xi_{0+})_x &= [\xi_{0+}, u], \\
(\xi_{i+})_x &= [\xi_{(i-1)+}, a] + [\xi_{i+}, u].
\end{aligned}
$$

As with the original derivation of the negative flows, we have $\xi_{0+} = m_{0+}^{-1}bm_{0+}$ and $m_{0+}^{-1}(m_{0+})_x = u$.

We have just proved the following theorem:

Theorem 8.1. *Let $a, b, f, E, M,$ and $\xi_{i\pm}$ be as above and $j > 0$. Recall that m_{1-} is the coefficient of λ^{-1} in M, then*

1. *the right dressing action of $e^{a\lambda x + b\lambda^j t}$ on f gives*
   - *(a) $\xi_{i-} = Q_{b,i}$ from theorem (4.1)*
   - *(b) $u = [a, m_{1-}]$ is a solution to the jth flow*
2. *the right dressing action of $e^{a\lambda x + b\lambda^{-j} t}$ on f gives*
   - *(a) $\xi_{i+} = \xi_i$ from equation (5.5)*
   - *(b) $u = [a, m_{1-}]$ is a solution to the $-j$th flow*

Theorem 8.2. *Each jth flow commutes with the others. In particular, the negative flows commute.*

The proof is similar to the discussion at the end of chapter 4. The idea is to show that the right dressing action of $\mathbb{L}^{\mathbb{R}}_{E}A$ on $\mathbb{L}^{\mathbb{R},\infty}_{I}SL_n\mathbb{C}$ induces an action on the phase space of the positive and negative flows (A is the space of diagonal matricies in $SL_n\mathbb{C}$). The proof differs from Terng's [5] due to the complex nature of the loop groups.

The phase space of the positive and negative flows is $C^{\infty}_{0}(\mathbb{R}, \mathfrak{sl}_n\mathbb{C}^{\perp}_{a})$ in the $SL_n\mathbb{C}$ hierarchy. Take $f \in \mathbb{L}^{\mathbb{R},\infty}_{I}SL_n\mathbb{C}$, define

$$m_f(x) = f *_r e^{a\lambda x}.$$

On a technical note, we look at the space of all germs of smooth maps from $\mathbb{R}$ to $\mathfrak{sl}_n\mathbb{C}^{\perp}_{a}$ and denote it $C_0(\mathbb{R}, \mathfrak{sl}_n\mathbb{C}^{\perp}_{a})$. Define the map

$$\Psi : \mathbb{L}^{\mathbb{R},\infty}_{I}SL_n\mathbb{C} \to C^{\infty}_{0}(\mathbb{R}, \mathfrak{sl}_n\mathbb{C}^{\perp}_{a})$$

by

$$\Psi(f) = [a, (m_f)_{1-}],$$

where $m_f(x) = ((m_f)_{0+} + (m_f)_{1+}\lambda + \cdots, I + (m_f)_{1-}\lambda^{-1} + \cdots)$. To show that the right dressing action of $\mathbb{L}^{\mathbb{R}}_{E}A$ on $\mathbb{L}^{\mathbb{R},\infty}_{I}SL_n\mathbb{C}$ induces an action on $\mathrm{Im}(\Psi)$, we first show that $\mathrm{Im}(\Psi)$ is the homogenous space $L^{\mathbb{R}}_{+}SL_n\mathbb{C} \times L^{\mathbb{R},\infty}_{-}SL_n\mathbb{C}/L^{\mathbb{R}}_{+}SL_n\mathbb{C} \times L^{\mathbb{R},\infty}_{-}A$. The first part of the product is defined on I_1 and the second on I_2.

Proposition 8.1. *1. If $f \in \mathbb{L}^{\mathbb{R},\infty}_{I}SL_n\mathbb{C}$, then $\Psi(f) = 0$ if and only if $f \in L_{+}SL_n\mathbb{C} \times L_{-}A$.*

2. If $f, g \in \mathbb{L}^{\mathbb{R},\infty}_{I}SL_n\mathbb{C}$, then $\Psi(f) = \Psi(g)$ if and only if $fg^{-1} \in L^{\mathbb{R}}_{+}SL_n\mathbb{C} \times L^{\mathbb{R},\infty}_{-}A$.

Proof. (1) If $f = (f_1, f_2) \in L_+^{\mathbb{R}} SL_n\mathbb{C} \times L_-^{\mathbb{R},\infty} A$ then

$$\begin{aligned}(f_1^{-1}, f_2^{-1})(e^{a\lambda x}, e^{a\lambda x}) &= (f_1^{-1}e^{a\lambda x}, f_2^{-1}e^{a\lambda x}) \\ &= (f_1^{-1}e^{a\lambda x}, e^{a\lambda x} f_2^{-1}) \\ &= \underbrace{(e^{a\lambda x}, e^{a\lambda x})}_{E} \underbrace{(e^{-a\lambda x} f_1^{-1} e^{a\lambda x}, f_2^{-1})}_{m^{-1}}.\end{aligned}$$

The decomposition is unique, so we now look at $(m_f)_{1-}(x)$ and since $f_2 a = a f_2$ we get $\Psi(f) = 0$. Conversely, if $\Psi(f) = 0$ then by theorem (4.1), $(m_f)_{j-} = 0$ for all j. This implies that $(m_1^{-1} a m_1, m_2^{-1} a m_2) = (m_1^{-1} a m_1, a)$ for $m = (m_1, m_2)$. Since a has distinct eigenvalues, $m_2(x)$ must be diagonal for all x, in particular, for $f_2 = m_2(0)$. Therefore, $f \in L_+^{\mathbb{R}} SL_n\mathbb{C} \times L_-^{\mathbb{R},\infty} A$.

(2) Suppose $\Psi(f) = \Psi(g) = u$ and $f^{-1}e^{a\lambda x} = E(x)m^{-1}(x), g^{-1}e^{a\lambda x} = F(x)n^{-1}(x)$. If we expand m and n in λ, we get

$$\begin{aligned}m(x)(\lambda) &= (m_{0+} + m_{1+}\lambda + \cdots, I + m_{1-}\lambda^{-1} + \cdots), \\ n(x)(\lambda) &= (n_{0+} + n_{1+}\lambda + \cdots, I + n_{1-}\lambda^{-1} + \cdots).\end{aligned}$$

Therefore, $u = [a, m_{1-}] = [a, n_{1-}]$. From the proof of (8.1), E and F satisfy

$$E^{-1}E_x = a\lambda + u, \quad F^{-1}F_x = a\lambda + u.$$

However, $E(0, \lambda) = F(0, \lambda) = I$. So E and F solve the same ODE and agree at a point and therefore $E = F$. More importantly, we have

$$F = f^{-1}e^{a\lambda x}m = g^{-1}e^{a\lambda x}n.$$

Rearranging the terms, we get

$$fg^{-1}e^{a\lambda x} = e^{a\lambda x}mn^{-1}.$$

Finally, we compute the $(mn^{-1})_{1-}$ term to be $m_{1-} - n_{1-}$. This can be seen by recalling the expansion for m and n in equation (8) and that $(mn^{-1})_{1-}$ is the coefficient of λ^{-1} in mn^{-1}. This allows us to compute

$$\Psi(fg^{-1}) = [a, m_{1-} - n_{1-}] = [a, m_{1-}] - [a, n_{1-}] = \Psi(f) - \Psi(g) = 0.$$

By part (1), we have $fg^{-1} \in L_+^{\mathbb{R}} SL_n\mathbb{C} \times L_-^{\mathbb{R},\infty} A$.

$\square$

One consequence of the proposition is that $\mathbb{L}_E^{\mathbb{R}} A$ acts on $Im(\Psi)$. This action is induced by the right dressing action as follows: Let $h \in \mathbb{L}_E^{\mathbb{R}} A$ and f as above, then

$$h * \Psi(f) \equiv \Psi(f *_r h).$$

If we let $h = e^{b\lambda^j t}$ for $j \in \mathbb{Z} - \{0\}$, then by theorem (8.1), $u = e^{b\lambda^j t} * \Psi(f)$ is a solution to the jth flow. Since $\mathbb{L}_E^{\mathbb{R}} A$ is abelian, the flows commute and theorem (8.2) is proven.

Chapter 9

The Harmonic Gauge

In the previous chapter, we saw that the right action of $e^{a\lambda x+b\lambda^j t}$ on $f \in \mathbb{L}_I^{\mathbb{R},\infty}G$ induces a solution to the jth flow. If we set $j=-1$ and let $H = F(x,t,\lambda)F(x,t,1)^{-1}$, then $H(x,t,\lambda)$ is a solution to the harmonic map equation (2.3). This is the same technique used in chapter (5). This is a gauge change which normalizes the extended harmonic map to be the identity at $\lambda = 1$, i.e. $H(x,t,1) = Id$. We call this the harmonic map gauge.

In this section, we will start in the harmonic map gauge and show that the techniques of the last section give rise to the harmonic map flow and its higher flows. Since, $1 \in E$, we must use the groups $\mathbb{L}_I^{\mathbb{R}}G_{\mathbb{C}}$ and $\mathbb{L}_E^{\mathbb{R},1}G_{\mathbb{C}}$ where

$$\mathbb{L}_E^{\mathbb{R},1}G_{\mathbb{C}} = \{f \in \mathbb{L}_E^{\mathbb{R}}G_{\mathbb{C}} \mid f(1) = Id\}.$$

Just as before, we will need to determine the explicit form of the splitting at the Lie algebra level. Given $f = (f_1, f_2) \in \mathbb{L}^{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$, we use the following modified formulas:

$$\lambda \in E: \; f_E(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-\lambda}dz - \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-1}dz,$$
$$\lambda \in I_1: \; f_{I_1}(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-1}dz,$$
$$\lambda \in I_2: \; f_{I_2}(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-1}dz.$$

Theorem 9.1.

$$f_E = f_{1_-} + f_{2_+} - (f_{1_-}(1) + f_{2_+}(1))$$
$$f_I = \big(f_{1_+} - f_{2_+} + (f_{1_-}(1) + f_{2_+}(1)), f_{2_-} - f_{1_-} + (f_{1_-}(1) + f_{2_+}(1))\big)$$

Proof. The only new quantity that needs to be calculated is the following term:

$$\begin{aligned}&\frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z-1}dz \\ =&\frac{1}{2\pi i}\int_{\gamma_1} \frac{g(z)}{z-1}dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{g(z)}{z-1}dz \\ =&f_{1_-}(1) + f_{2_+}(1).\end{aligned}$$

□

This new projection moves the elements constant in λ to the I algebra. For instance, let $g = (C, D)$, where C and D are constant in λ. Then by theorem (9.1) we have

$$g_E = (C, D)_E = (D - D, D - D) = (0, 0)$$
$$g_I = (C, D)_I = (C - D + D, D) = (C, D).$$

Again, let $G = SL_n\mathbb{C}$ and $a, b \in \mathfrak{su}_n \subset \mathfrak{sl}_n\mathbb{C}$ be regular diagonal elements. Let $f(\lambda) \in \mathbb{L}_I^{\mathbb{R}} SL_n\mathbb{C}$, then

$$f(\lambda) *_r e^{a(\lambda-1)x + b(\lambda^{-j}-1)t} = M(x,t,\lambda)^{-1} \text{ where,}$$
$$f(\lambda)^{-1} e^{a(\lambda-1)x + b(\lambda^{-j}-1)t} = F(x,t,\lambda)M(x,t,\lambda)^{-1}.$$

Where $F \in \mathbb{L}_E^{\mathbb{R},1} SL_n\mathbb{C}$, $M \in \mathbb{L}_I^{\mathbb{R}} SL_n\mathbb{C}$ and $j > 0$. The next step is to compute the equations:

$$\begin{aligned} F^{-1}F_x =& (M^{-1}a(\lambda - 1)M)_E, \\ F^{-1}F_t =& (M^{-1}b(\lambda^{-j} - 1)M)_E. \end{aligned} \tag{9.1}$$

We can simplify (9.1) by writing

$$M = (m_{0+} + m_{1+}\lambda + m_{2+}\lambda^2 + \cdots,\ m_{0-} + m_{1-}\lambda^{-1} + m_{2-}\lambda^{-2} + \cdots)$$

and applying theorem (9.1) to $(M^{-1}aM)_E$. We get

$$\begin{aligned} (M^{-1}aM)_E &= (m_{0+}^{-1}am_{0+} + \cdots, m_{0-}^{-1}am_{0-} + \cdots)_E \\ &= (m_{0+}^{-1}am_{0+} - m_{0+}^{-1}am_{0+}, m_{0-}^{-1}am_{0-} - m_{0-}^{-1}am_{0-}) \\ &= 0. \end{aligned}$$

Therefore, (9.1) reduces to

$$\begin{aligned} F^{-1}F_x =& (M^{-1}a\lambda M)_E, \\ F^{-1}F_t =& (M^{-1}b\lambda^{-j}M)_E. \end{aligned} \tag{9.2}$$

In order to compute the flows in equation (9.2), we will make use of the following notation:

$$M^{-1}XM = (\xi_{0+}^X + \xi_{1+}^X\lambda + \xi_{2+}^X\lambda^2 + \cdots, \xi_{0-}^X + \xi_{1-}^X\lambda^{-1} + \xi_{2-}^X\lambda^{-2} + \cdots).$$

For $X = a\lambda$ or $b\lambda^{-j}$, we get:

$$
\begin{aligned}
F^{-1}F_x &= (M^{-1}a\lambda M)_E \\
&= (\xi^a_{0+}\lambda + \xi^a_{1+}\lambda^2 + \xi^a_{2+}\lambda^3 + \cdots, \xi^a_{0-}\lambda + \xi^a_{1-} + \xi^a_{2-}\lambda^{-1} + \cdots)_E \\
&= \xi^a_{0-}\lambda - \xi^a_{0-} \\
&= \xi^a_{0-}(\lambda - 1). \\
F^{-1}F_t &= (M^{-1}b\lambda^{-j}M)_E \\
&= (\xi^b_{0+}\lambda^{-j} + \xi^b_{1+}\lambda^{-j+1} + \cdots, \xi^b_{0-}\lambda^{-j} + \xi^b_{1-}\lambda^{-j-1} + \cdots)_E \\
&= \xi^b_{0+}\lambda^{-j} + \xi^b_{1+}\lambda^{-j+1} + \cdots + \xi^b_{j-1}\lambda^{-1} - \xi^b_{0+} + \xi^b_{1+} + \cdots + \xi^b_{j-1} \\
&= \xi^b_{0+}(\lambda^{-j} - 1) + \xi^b_{1+}(\lambda^{-j+1} - 1) + \cdots + \xi^b_{j-1}(\lambda^{-1} - 1).
\end{aligned}
\tag{9.3}
$$

If $j = 1$ then equation (9.2) reduces to the harmonic map equation. Also, this can be done for $-j > 0$ to give the positive flows in the harmonic gauge.

Now we have two ways to obtain a solution in the harmonic gauge. The first is to use the methods of the last chapter, the loop groups $\mathbb{L}^{\mathbb{R}}_E G$ and $\mathbb{L}^{\mathbb{R},\infty}_I G$, and to make a gauge change. The second is to start in the harmonic map gauge by using the loop groups $\mathbb{L}^{\mathbb{R},1}_E G$ and $\mathbb{L}^{\mathbb{R}}_I G$.

One question might be, "Are these flows the same?". In other words, are the flows obtained by applying the right dressing action of $e^{a\lambda x + b\lambda^j t}$ on$f \in \mathbb{L}^{\mathbb{R},\infty}_I$ the same as the flows obtained by the right dressing action of $e^{a(\lambda-1)x+b(\lambda^j-1)t}$ on $f \in \mathbb{L}^{\mathbb{R},\infty}_i$? All that needs to be shown is that $\tilde{F}(x,t,\lambda) = F(x,t,\lambda)F(x,t,1)^{-1}$. Here $\tilde{F}$ is obtained from the harmonic gauge and F is obtained from the techniques of chapter (8). This is shown below.

$$\overbrace{f^{-1}}^{\mathbb{L}_I^{\mathbb{R},\infty}G}\;\overbrace{e^{a\lambda x+b\lambda^j t}}^{\mathbb{L}_E^{\mathbb{R}}G} = \overbrace{F(x,t,\lambda)}^{\mathbb{L}_E^{\mathbb{R}}G}\,\overbrace{M(x,t,\lambda)^{-1}}^{\mathbb{L}_I^{\mathbb{R},\infty}G}$$

$$\Leftrightarrow$$

$$f^{-1}e^{a(\lambda-1)x+b(\lambda^j-1)t}e^{ax+bt} = F(x,t,\lambda)F(x,t,1)^{-1}F(x,t,1)M(x,t,\lambda)$$

$$\Leftrightarrow$$

$$\underbrace{f^{-1}}_{\mathbb{L}_I^{\mathbb{R}}G}\,\underbrace{e^{a(\lambda-1)x+b(\lambda^j-1)t}}_{\mathbb{L}_E^{\mathbb{R},1}G} = \underbrace{F(x,t,\lambda)F(x,t,1)^{-1}}_{\mathbb{L}_E^{\mathbb{R},1}G}\,\underbrace{F(x,t,1)M(x,t,\lambda)e^{ax+bt}}_{\mathbb{L}_I^{\mathbb{R}}G}.$$

Therefore, since the factorization is unique when it exists, the flows are the same.

Some final remarks about this method. First, if we want to recreate the exact form of our connections A_λ, B_λ from chapter 2, we only need to apply the right action of $e^{\frac{1}{2}a(1-\lambda)x+\frac{1}{2}b(1-\lambda^{-j})t}$, for $j=1$ instead of $e^{a(\lambda-1)x+b(\lambda^{-1}-1)t}$.

Second, it is possible to obtain equations for the coefficients of λ in

$$M^{-1}b(\lambda-1)M = (\xi^b_{0+}\lambda^{-j} + \xi^b_{1+}\lambda^{-j+1} + \cdots, \xi^b_{0-}\lambda^{-j} + \xi^b_{1-}\lambda^{-j-1} + \cdots)$$

just as we did for the negative flows in equation (5.3). First, by equation (8.2) (which still holds in the harmonic gauge) we have

$$(M^{-1}bM)_x = [a(\lambda-1), M^{-1}bM]. \tag{9.4}$$

This gives the following equations:

$$(\xi^b_{0+})_x + [\xi^b_{0+}, a] = 0$$
$$(\xi^b_{i+})_x + [\xi^b_{i+}, a] + [a, \xi^b_{(i-1)+}] = 0,$$
$$(\xi^b_{i-})_x + [\xi^b_{i-}, a] + [a, \xi^b_{(i+1)-}] = 0.$$

Next, we have

$$\xi_{0+} = h^{-1}bh, \text{ where}$$
$$h^{-1}h_x = a.$$

Finally, we get that

$$\xi_{i+}^b(x) = -h^{-1}(x)\left(\int_\infty^x h(y)[a, \xi_{(i-1)+}^b]h^{-1}(y)dy\right)h(x),$$
$$\xi_{i-}^b(x) = -h^{-1}(x)\left(\int_\infty^x h(y)[a, \xi_{(i+1)-}^b]h^{-1}(y)dy\right)h(x).$$

Chapter 10

Classes of Harmonic Maps

In this chapter we try to expand the classes of harmonic maps that are included in the -1-flow theory. The goal will be to replace constants a and b with smooth functions $\alpha(x)$ and $\beta(t)$. The claim is this change does not affect the theory and we obtain a substantially larger set of harmonic maps.

Let $\alpha(x) \in A$ for all x and $\beta(t) \in A$ for all t. Recall that A is the set of all diagonal matricies in $\mathfrak{su}_n$. Next, we look at the right action of $e^{\alpha(x)\lambda+\beta(t)\lambda^{-1}}$ on f.

$$\begin{aligned} f *_r e^{\alpha(x)\lambda+\beta(t)\lambda^{-1}} &= M(x,t,\lambda) \text{ where,} \\ f^{-1}e^{\alpha(x)\lambda+\beta(t)\lambda^{-1}} &= F(x,t,\lambda)M(x,t,\lambda)^{-1}. \end{aligned}$$

Next, we calculate $F^{-1}F_x$ and $F^{-1}F_t$.

$$\begin{aligned} F^{-1}F_x &= (M^{-1}e^{-\alpha(x)\lambda-\beta(t)\lambda^{-1}}f^{-1})(f\alpha'(x)e^{\alpha(x)\lambda+\beta(t)\lambda^{-1}}M \\ &\quad - fe^{\alpha(x)\lambda+\beta(t)\lambda^{-1}}M^{-1}M_xM^{-1} \\ &= M^{-1}\alpha'(x)\lambda M - M_xM^{-1} \\ &= (M^{-1}\alpha'(x)\lambda M)_E \\ &= \alpha'(x)\lambda + [\alpha'(x), m_{1-}]. \end{aligned}$$

$$
\begin{aligned}
F^{-1}F_t &= (M^{-1}e^{-\alpha(x)\lambda-\beta(t)\lambda^{-1}}f^{-1})(f\beta'(t)e^{\alpha(x)\lambda+\beta(t)\lambda^{-1}}M) \\
&\quad - fe^{\alpha(x)\lambda+\beta(t)\lambda^{-1}}M^{-1}M_tM^{-1} \\
&= M^{-1}\beta'(t)\lambda M - M_xM^{-1} \\
&= (M^{-1}\beta'(t)\lambda M)_E \\
&= g^{-1}\beta'(t)g\lambda^{-1}.
\end{aligned}
$$

Theorem 10.1. *$\phi(x,t) \equiv F(x,t,-1)F^{-1}(x,t,1)$ is a harmonic map.*

Proof. The proof is a straight forward calculation. We must check that ϕ satisfies

$$(\phi^{-1}\phi_x)_t + (\phi^{-1}\phi_t)_x = 0.$$

This is done by substituting the definition of ϕ into the equation. Let $F_1 = F(x,t,1)$ and $F_{-1} = F(x,t,-1)$. Then we have

$$
\begin{aligned}
(\phi^{-1}\phi_x)_t &= \big((F_1F_{-1}^{-1})(F_{-1x}F_1^{-1} - F_{-1}E_1^{-1}F_{1x}F_1^{-1})\big)_t \\
&= (F_1((u-\alpha') - (u+\alpha'))F_1^{-1})_t \\
&= -2(F_1\alpha'F_1^{-1})_t \\
&= -2(F_{1t}\alpha'F_1^{-1} - F_1\alpha'F_1^{-1}F_{1t}F_1^{-1}) \\
&= -2F_1(g^{-1}\beta'g\alpha' - \alpha'g^{-1}\beta'g)F_1^{-1} \\
&= 2F_1([\alpha', g^{-1}\beta'g])F_1^{-1}.
\end{aligned}
$$

$$
\begin{aligned}
(\phi^{-1}\phi_t)_x &= \big((F_1F_{-1}^{-1})(F_{-1t}F_1^{-1} - F_{-1}F_1^{-1}F_{1t}F_1^{-1})\big)_x \\
&= (F_1(-g^{-1}\beta' g - g^{-1}\beta' g)F_1^{-1})_x \\
&= -2(F_1 g^{-1}\beta' g F_1^{-1})_x \\
&= -2(F_{1x} g^{-1}\beta' g F_1^{-1} + F_1(-g^{-1}g_x g^{-1}\beta' g + g^{-1}\beta' g_x)F_1^{-1} \\
&\quad - F_1 g^{-1}\beta' g F_1^{-1} F_{1x} F_1^{-1}) \\
&= -2F_1\left((\alpha' + u)g^{-1}\beta' g + [g^{-1}\beta' g, u] - g^{-1}\beta' g(a+u)\right)F_1^{-1} \\
&= -2F_1[\alpha', g^{-1}\beta' g]F_1^{-1}.
\end{aligned}
$$

□

Next, recall that given a harmonic map ϕ, we have the lax pair:

$$(A_\lambda)_\eta - (B_\lambda)_\xi = [A_\lambda, B_\lambda],$$

where $\phi^{-1}\phi_\xi = A$, $\phi^{-1}\phi_\eta = B$, $B_\lambda = \frac{1-\lambda^{-1}}{2}B$, and $A_\lambda = \frac{1-\lambda}{2}A$. One question is, "What do A and B look like when we make a gauge transformation from $a\lambda + u \to A_\lambda$ and $g^{-1}bg \to B_\lambda$?"

If we let $h(x,t)$ be our gauge change, then we have

$$h\left(\frac{\partial}{\partial \xi} + a\lambda + u\right)h^{-1} = \frac{\partial}{\partial \xi} + hah^{-1}\lambda + huh^{-1} - h_x h^{-1}.$$

To normalize at $\lambda = 1$, we need

$$h_x h^{-1} = huh^{-1} - hah^{-1}$$

or

$$h^{-1}h_x = u - a.$$

Regardless of how we write it, we get $A = -hah^{-1}$ and $B = -hg^{-1}bgh^{-1}$. This says that $\phi^{-1}\phi_\xi$ resides in the class of a for all η and ξ. Also, we have that $\phi^{-1}\phi_\eta$ resides in the conjugacy class of b for all η and ξ.

From the theory of Lie groups, we know that every element of a compact Lie algebra can be conjugated into the Lie algebra of the maximal torus. However, we have been assuming that a and b have distinct eigen-values.

Also, if we use the theory from the beginning of this chapter, we get that $A = -h\alpha(\xi)h^{-1}$ and $B = -hg^{-1}\beta(\eta)gh^{-1}$. This means if we restrict to SU_n, we obtain all harmonic maps that do not intersect conjugacy classes of torus elements with non-distinct eigenvalues.

Chapter 11

Application to the Virasoro Action

In this chapter, we examine how changes to $f \in \mathbb{L}_I^{\mathbb{R},\infty} G_{\mathbb{C}}$ (traditionally called the scattering data) affect the term $A = a\lambda + u$. Once this is determined it can be applied to various actions on the regions I_1 and I_2.

Theorem 11.1. $\delta A = [A, (F^{-1} f^{-1} \delta f F)_I]_E$.

Proof. First, we note that

$$\begin{aligned} f^{-1} e^{a\lambda x + b\lambda^j t} &= FM^{-1} \\ A =& F^{-1} F_x. \end{aligned}$$

By taking the differential of each equation with respect to a change in f we get

$$\begin{aligned} -f^{-1} \delta f f^{-1} e^{a\lambda x + b\lambda^j t} &= \delta F M^{-1} - FM^{-1} \delta M M^{-1} \\ \delta A &= -F^{-1} \delta F F^{-1} F_x + F^{-1} \delta (F_x). \end{aligned}$$

The first equation can be manipulated to get

$$-F^{-1}f^{-1}\delta f F = F^{-1}\delta F - M^{-1}\delta M,$$

which gives

$$(-F^{-1}f^{-1}\delta f F)_E = F^{-1}\delta F. \tag{11.1}$$

Finally, we have

$$\begin{aligned}
\delta A &= -F^{-1}\delta F F^{-1}F_x + F^{-1}(\delta F)_x \\
&= (F^{-1}f^{-1}\delta f F)_E A + F^{-1}(F(-F^{-1}f^{-1}\delta f F)_E)_x \\
&= (F^{-1}f^{-1}\delta f F)_E A + F^{-1}(F_x(-F^{-1}f^{-1}\delta f F)_E + F((-F^{-1}f^{-1}\delta f F)_E)_x) \\
&= [(F^{-1}f^{-1}\delta f F)_E, A] + [A, F^{-1}f^{-1}\delta f F]_E \\
&= [A, (F^{-1}f^{-1}\delta f F)_I]_E.
\end{aligned}$$

The last line is due to the fact that

$$\begin{aligned}
[X_E, A] + [A, X]_E &= [X_E, A] + [A, X_E]_E + [A, X_I]_E \\
&= [X_E, A]_I + [A, X_I]_E,
\end{aligned}$$

and the first term of the last equation is zero. □

This result can be applied to the paper of Uhlenbeck and Vajiac [11]. In this paper, the derived action of the half-Virasoro algebra on the E group is examined. As before, we use the wave reality condition $F(\lambda)^{-1} = (F(\bar{\lambda}))^*$. Also, we use the harmonic gauge and the associated factorization of $\mathbb{L}^{\mathbb{R}}G_{\mathbb{C}}$ used in chapter 9.

The Virasoro algebra is the infinitesimal generators of diffeomorphisms of the circle. The basis of the Virasoro algebra are elements $L_j = \lambda^{j+1}\frac{\partial}{\partial\lambda}$. With this notation, we have the formula $[L_j, L_k] = (k-j)L_{j+k}$.

In the case of $\mathbb{L}^{\mathbb{R}}G_{\mathbb{C}}$, there are two circles to act on. Let

$$\begin{aligned} w(\lambda)\frac{\partial}{\partial\lambda} \in \mathbb{V}^{+}_{\mathbb{R},0} = &\ \operatorname{span}_{\mathbb{R}}\left\{L_0, L_1, \ldots, L_i, \ldots\right\}, \\ v(\lambda)\frac{\partial}{\partial\lambda} \in \mathbb{V}^{-}_{\mathbb{R},\infty} = &\ \operatorname{span}_{\mathbb{R}}\left\{\ldots, L_{-i}, \ldots, L_{-1}, L_0\right\}. \end{aligned}$$

In [11], The derived action of the element $V = (w(\lambda)\frac{\partial}{\partial\lambda}, v(\lambda)\frac{\partial}{\partial\lambda})$ on an extended harmonic map $F(x,t,\lambda)$ is shown to be

$$\begin{aligned} V^{\#}F_\lambda = &\frac{1}{2\pi i}F_\lambda\Big[\int_{|\gamma|=\epsilon}\frac{F_\gamma^{-1}w(\gamma)\frac{\partial}{\partial\gamma}F_\gamma(\lambda-1)}{(\lambda-\gamma)(\gamma-1)}d\gamma \\ &+\int_{|\gamma|=\epsilon^{-1}}\frac{F_\gamma^{-1}v(\gamma)\frac{\partial}{\partial\gamma}F_\gamma(\lambda-1)}{(\lambda-\gamma)(\gamma-1)}d\gamma\Big] \end{aligned} \tag{11.2}$$

However, in order to switch to the notation used in this paper, we use the method of partial fractions to rewrite the integrals. Also, we integrate around Γ, which is the union of the circles of radius ϵ and ϵ^{-1}. Thus, equation (11.2) becomes

$$\begin{aligned} V^{\#}F_\lambda = &\frac{1}{2\pi i}\Big[\int_\Gamma\frac{F_\gamma^{-1}(w(\gamma)\frac{\partial}{\partial\gamma}F_\gamma, v(\gamma)\frac{\partial}{\partial\gamma}F_\gamma)}{(\gamma-\lambda)}d\gamma \\ &-\int_\Gamma\frac{F_\gamma^{-1}(w(\gamma)\frac{\partial}{\partial\gamma}F_\gamma, v(\gamma)\frac{\partial}{\partial\gamma}F_\gamma)}{(\gamma-1)}d\gamma\Big] \end{aligned} \tag{11.3}$$

By theorem (9.1), we get

$$\begin{aligned} V^{\#}F &= F\left(F^{-1}(w(\lambda)\frac{\partial}{\partial\lambda}F, v(\lambda)\frac{\partial}{\partial\lambda}F)\right)_E \\ &= F\left(F^{-1}V(F)\right)_E. \end{aligned}$$

To reiterate the notation, an element $F \in \mathbb{L}_E^{\mathbb{R},1}G_{\mathbb{C}}$ is actually two maps and is more carefully written as $F = (F, F)$.

Next, the direct Virasoro action of V on an element $H = (H_1, H_2) \in \mathbb{L}^{\mathbb{R}}G_{\mathbb{C}}$

is defined as

$$V(H(\lambda)) = (w(\lambda)\frac{\partial}{\partial\lambda}H_1(\lambda), v(\lambda)\frac{\partial}{\partial\lambda}H_2(\lambda)).$$

Theorem 11.2. *For $w(\lambda) = \lambda^k$ and $v(\lambda) = \lambda^{-j}$, $k \geq 2$, $j \geq 0$ the formula for the derived Virasoro action equal to equation (11.1), where δf is the direct Virasoro action on $f \in \mathbb{L}_I^{\mathbb{R}}G_{\mathbb{C}}$. In other words,*

$$F^{-1}V^{\#}F = (-F^{-1}f^{-1}\delta f F)_E.$$

Proof. From equation (11.1) we have

$$\begin{aligned}
F^{-1}\delta F &= (-F^{-1}f^{-1}V(f)F)_E \\
&= (-F^{-1}f^{-1}V(fF) + F^{-1}f^{-1}fV(F))_E \\
&= (M^{-1}e^{-(a\lambda x+b\lambda^{-1}t)}V(e^{a\lambda x+b\lambda^{-1}t}M) + F^{-1}V(F))_E \\
&= (M^{-1}(a\lambda^k x - b\lambda^{k-2}t, a\lambda^{-j}x - b\lambda^{-j-2}t)M+ \\
&+ M^{-1}V(M) + F^{-1}V(F))_E
\end{aligned}$$

We need to show that the first two terms of the last equation are zero for $k \geq 2$. (The restriction $j \geq 0$ is satisfied because $v(\lambda)\frac{\partial}{\partial\lambda} \in \mathbb{V}^-_{\mathbb{R},\infty}$) First,

$$\begin{aligned}
M^{-1}V(M) &= M^{-1}(\lambda^k\frac{\partial}{\partial\lambda}, \lambda^{-j}\frac{\partial}{\partial\lambda})(M_1, M_2) \\
&= M^{-1}(\lambda^k\frac{\partial}{\partial\lambda}, \lambda^{-j}\frac{\partial}{\partial\lambda})(m_{0+} + m_{1+}\lambda + \cdots, m_{0-} + m_{1-}\lambda^{-1} + \cdots) \\
&= M^{-1}(m_{1+}\lambda^k + 2m_{2+}\lambda^{k+1} + \cdots, -m_{1-}\lambda^{-j-2} - 2m_{2-}\lambda^{-j-3} + \cdots).
\end{aligned}$$

Therefore, $(M^{-1}V(M))_E = 0$ as long as $k \geq 0$.

Next, we look at the term

$$P = (P_1, P_2) = M^{-1}(a\lambda^k x - b\lambda^{k-2}t, a\lambda^{-j}x - b\lambda^{-j-2}t)M.$$

Notice if $k \geq 2$, there are no terms with negative powers of λ in P_1 and there are no terms with positive powers of λ in P_2 for $j \geq 0$. Therefore, we have that

$$F^{-1}\delta F = F^{-1}V^{\#}F,$$

which proves the theorem. $\square$

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Vita

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